極座標

$$d\ell = r\,d\theta, \qquad dS = r\,d\theta\, r\sin\theta\, d\phi, \qquad dV = dS\,dr = r^2\sin\theta\, d\theta\, d\phi\, dr$$

面積 $$S = \int\int r^2 \sin\theta\, d\theta\, d\phi = r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi r^2$$

体積 $$V = \int\int r^2\, dr \sin\theta\, d\theta\, d\phi = 4\pi \int_0^{r} r^2\, dr = \frac{4}{3}\pi r^3$$

$$x = r\sin\theta\cos\phi \qquad (1) \qquad\qquad r^2 = x^2 + y^2 + z^2 \qquad (4)$$
$$y = r\sin\theta\sin\phi \qquad (2) \qquad\qquad \tan\phi = y/x \qquad (5)$$
$$z = r\cos\theta \qquad (3) \qquad\qquad \cos\theta = z/r \qquad (6)$$

偏微分の連鎖規則

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,\phi}\left(\frac{\partial r}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial \theta}\right)_{r,\phi}\left(\frac{\partial \theta}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial \phi}\right)_{r,\theta}\left(\frac{\partial \phi}{\partial x}\right)_{y,z}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,\phi}\left(\frac{\partial r}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial \theta}\right)_{r,\phi}\left(\frac{\partial \theta}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial \phi}\right)_{r,\theta}\left(\frac{\partial \phi}{\partial y}\right)_{x,z}$$

$$\left(\frac{\partial f}{\partial z}\right)_{x,y} = \left(\frac{\partial f}{\partial r}\right)_{\theta,\phi}\left(\frac{\partial r}{\partial z}\right)_{x,y} + \left(\frac{\partial f}{\partial \theta}\right)_{r,\phi}\left(\frac{\partial \theta}{\partial z}\right)_{x,y} + \left(\frac{\partial f}{\partial \phi}\right)_{r,\theta}\left(\frac{\partial \phi}{\partial z}\right)_{x,y}$$

(1),(2),(3) 式の両辺を x で微分する（平面極座標参照）

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial r\sin\theta\cos\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial r\sin\theta\cos\phi}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial r\sin\theta\cos\phi}{\partial \phi}\frac{\partial \phi}{\partial x}$$
$$= \sin\theta\cos\phi\frac{\partial r}{\partial x} + r\cos\theta\cos\phi\frac{\partial \theta}{\partial x} + r\sin\theta(-\sin\phi)\frac{\partial \phi}{\partial x}$$

$$\frac{\partial y}{\partial x} = 0 = \frac{\partial r\sin\theta\sin\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial r\sin\theta\sin\phi}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial r\sin\theta\sin\phi}{\partial \phi}\frac{\partial \phi}{\partial x}$$
$$= \sin\theta\sin\phi\frac{\partial r}{\partial x} + r\cos\theta\sin\phi\frac{\partial \theta}{\partial x} + r\sin\theta\cos\phi\frac{\partial \phi}{\partial x}$$

$$\frac{\partial z}{\partial x} = 0 = \frac{\partial r\cos\theta}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial r\cos\theta}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial r\cos\theta}{\partial \phi}\frac{\partial \phi}{\partial x}$$
$$= \cos\theta\frac{\partial r}{\partial x} - r\sin\theta\frac{\partial \theta}{\partial x}$$

$$D=\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & r\sin\theta(-\sin\phi) \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$=r^2(\sin\theta\cos^2\theta\cos^2\phi+\sin^3\theta\sin^2\phi$$

$$+\sin^3\theta\cos^2\phi+\sin\theta\cos^2\theta\sin^2\phi)$$

$$=r^2(\sin\theta\cos^2\theta+\sin^3\theta)=r^2\sin\theta$$

$$\frac{\partial r}{\partial x}=\begin{vmatrix} 1 & r\cos\theta\cos\phi & r\sin\theta(-\sin\phi) \\ 0 & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & -r\sin\theta & 0 \end{vmatrix}/D$$

$$=r^2\sin^2\theta\cos\phi/r^2\sin\theta=\sin\theta\cos\phi$$

$$\frac{\partial\theta}{\partial x}=\begin{vmatrix} \sin\theta\cos\phi & 1 & r\sin\theta(-\sin\phi) \\ \sin\theta\sin\phi & 0 & r\sin\theta\cos\phi \\ \cos\theta & 0 & 0 \end{vmatrix}/D$$

$$=r\sin\theta\cos\theta\cos\phi/r^2\sin\theta=\cos\theta\cos\phi/r$$

$$\frac{\partial\phi}{\partial x}=\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & 1 \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}/D$$

$$=r(-\cos^2\theta\sin\phi-\sin^2\theta\sin\phi)/r^2\sin\theta=\frac{-\sin\phi}{r\sin\theta}$$

同様にしてｙでの微分を行う。

$$\frac{\partial x}{\partial y}=0=\frac{\partial r\sin\theta\cos\phi}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial r\sin\theta\cos\phi}{\partial\theta}\frac{\partial\theta}{\partial y}+\frac{\partial r\sin\theta\cos\phi}{\partial\phi}\frac{\partial\phi}{\partial y}$$

$$=\sin\theta\cos\phi\frac{\partial r}{\partial y}+r\cos\theta\cos\phi\frac{\partial\theta}{\partial y}+r\sin\theta(-\sin\phi)\frac{\partial\phi}{\partial y}$$

$$\frac{\partial y}{\partial y}=1=\frac{\partial r\sin\theta\sin\phi}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial r\sin\theta\sin\phi}{\partial\theta}\frac{\partial\theta}{\partial y}+\frac{\partial r\sin\theta\sin\phi}{\partial\phi}\frac{\partial\phi}{\partial y}$$

$$=\sin\theta\sin\phi\frac{\partial r}{\partial y}+r\cos\theta\sin\phi\frac{\partial\theta}{\partial y}+r\sin\theta\cos\phi\frac{\partial\phi}{\partial y}$$

$$\frac{\partial z}{\partial y}=0=\frac{\partial r\cos\theta}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial r\cos\theta}{\partial \theta}\frac{\partial \theta}{\partial y}+\frac{\partial r\cos\theta}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$=\cos\theta\frac{\partial r}{\partial y}-r\sin\theta\frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial y}=\begin{vmatrix} 0 & r\cos\theta\cos\phi & r\sin\theta(-\sin\phi) \\ 1 & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & -r\sin\theta & 0 \end{vmatrix}/D$$

$$=r^{2}\sin^{2}\theta\sin\phi/r^{2}\sin\theta=\sin\theta\sin\phi$$

$$\frac{\partial \theta}{\partial y}=\begin{vmatrix} \sin\theta\cos\phi & 0 & r\sin\theta(-\sin\phi) \\ \sin\theta\sin\phi & 1 & r\sin\theta\cos\phi \\ \cos\theta & 0 & 0 \end{vmatrix}/D$$

$$=r\sin\theta\cos\theta\sin\phi\ /\ r^{2}\sin\theta=\cos\theta\sin\phi\ /\ r$$

$$\frac{\partial \phi}{\partial y}=\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & 0 \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & 1 \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}/D$$

$$=r(\cos^{2}\theta\cos\phi+\sin^{2}\theta\cos\phi)/r^{2}\sin\theta=\frac{\cos\phi}{r\sin\theta}$$

zでの微分を行う。

$$\frac{\partial x}{\partial z}=0=\frac{\partial r\sin\theta\cos\phi}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial r\sin\theta\cos\phi}{\partial \theta}\frac{\partial \theta}{\partial z}+\frac{\partial r\sin\theta\cos\phi}{\partial \phi}\frac{\partial \phi}{\partial z}$$

$$=\sin\theta\cos\phi\frac{\partial r}{\partial z}+r\cos\theta\cos\phi\frac{\partial \theta}{\partial z}+r\sin\theta(-\sin\phi)\frac{\partial \phi}{\partial z}$$

$$\frac{\partial y}{\partial z}=0=\frac{\partial r\sin\theta\sin\phi}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial r\sin\theta\sin\phi}{\partial \theta}\frac{\partial \theta}{\partial z}+\frac{\partial r\sin\theta\sin\phi}{\partial \phi}\frac{\partial \phi}{\partial z}$$

$$=\sin\theta\sin\phi\frac{\partial r}{\partial z}+r\cos\theta\sin\phi\frac{\partial \theta}{\partial z}+r\sin\theta\cos\phi\frac{\partial \phi}{\partial z}$$

$$\frac{\partial z}{\partial z}=1=\frac{\partial r\cos\theta}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial r\cos\theta}{\partial \theta}\frac{\partial \theta}{\partial z}+\frac{\partial r\cos\theta}{\partial \phi}\frac{\partial \phi}{\partial z}$$

$$=\cos\theta\frac{\partial r}{\partial z}-r\sin\theta\frac{\partial \theta}{\partial z}$$

$$\frac{\partial r}{\partial z} = \begin{vmatrix} 0 & r\cos\theta\cos\phi & r\sin\theta(-\sin\phi) \\ 0 & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 1 & -r\sin\theta & 0 \end{vmatrix} / D$$

$$= r^2 (\cos\theta\sin\theta\sin^2\phi + \cos\theta\sin\theta\cos^2\phi) / r^2 \sin\theta = \cos\theta$$

$$\frac{\partial \theta}{\partial z} = \begin{vmatrix} \sin\theta\cos\phi & 0 & r\sin\theta(-\sin\phi) \\ \sin\theta\sin\phi & 0 & r\sin\theta\cos\phi \\ \cos\theta & 1 & 0 \end{vmatrix} / D$$

$$= -r(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) / r^2 \sin\theta = -\sin\theta / r$$

$$\frac{\partial \phi}{\partial z} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & 0 \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ \cos\theta & -r\sin\theta & 1 \end{vmatrix} / D$$

$$= r(\sin\theta\cos\phi\cos\theta\sin\phi - \sin\theta\sin\phi\cos\theta\cos\phi) / r^2 \sin\theta = 0$$

結果をまとめると

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\sin\theta\cos\phi + \frac{\partial f}{\partial \theta}\frac{\cos\theta\cos\phi}{r} - \frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\sin\theta\sin\phi + \frac{\partial f}{\partial \theta}\frac{\cos\theta\sin\phi}{r} + \frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}$$

もう一度偏微分の連鎖規則を適応して２階微分を求める

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)\frac{\partial \theta}{\partial x}+\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial x}\right)\frac{\partial \phi}{\partial x}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\frac{\cos\theta\cos\phi}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right)\sin\theta\cos\phi$$

$$+\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\frac{\cos\theta\cos\phi}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right)\frac{\cos\theta\cos\phi}{r}$$

$$+\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\frac{\cos\theta\cos\phi}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right)\frac{-\sin\phi}{r\sin\theta}$$

$$=\left\{\frac{\partial^2 f}{\partial r^2}\sin\theta\cos\phi+\frac{\partial^2 f}{\partial r\partial \theta}\frac{\cos\theta\cos\phi}{r}+\frac{\partial f}{\partial \theta}\frac{-\cos\theta\cos\phi}{r^2}\right.$$

$$\left.-\frac{\partial^2 f}{\partial r\partial \phi}\frac{\sin\phi}{r\sin\theta}+\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r^2\sin\theta}\right\}\sin\theta\cos\phi+\left\{\frac{\partial^2 f}{\partial \theta\partial r}\sin\theta\cos\phi\right.$$

$$+\frac{\partial f}{\partial r}\cos\theta\cos\phi+\frac{\partial^2 f}{\partial \theta^2}\frac{\cos\theta\cos\phi}{r}+\frac{\partial f}{\partial \theta}\frac{-\sin\theta\cos\phi}{r}$$

$$\left.-\frac{\partial^2 f}{\partial \theta\partial \phi}\frac{\sin\phi}{r\sin\theta}-\frac{\partial f}{\partial \phi}\frac{-\cos\theta\sin\phi}{r\sin^2\theta}\right\}\frac{\cos\theta\cos\phi}{r}+\left\{\frac{\partial^2 f}{\partial \phi\partial r}\sin\theta\cos\phi\right.$$

$$+\frac{\partial f}{\partial r}\sin\theta(-\sin\phi)+\frac{\partial^2 f}{\partial \phi\partial \theta}\frac{\cos\theta\cos\phi}{r}+\frac{\partial f}{\partial \theta}\frac{-\cos\theta\sin\phi}{r}$$

$$\left.-\frac{\partial^2 f}{\partial \phi^2}\frac{\sin\phi}{r\sin\theta}-\frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}\right\}\frac{-\sin\phi}{r\sin\theta}$$

$$=\frac{\partial^2 f}{\partial r^2}\sin^2\theta\cos^2\phi+\frac{\partial^2 f}{\partial r\partial \theta}\frac{\sin\theta\cos\theta\cos^2\phi}{r}-\frac{\partial f}{\partial \theta}\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}$$

$$-\frac{\partial^2 f}{\partial r\partial \phi}\frac{\sin\phi\cos\phi}{r}+\frac{\partial f}{\partial \phi}\frac{\sin\phi\cos\phi}{r^2}+\frac{\partial^2 f}{\partial \theta\partial r}\frac{\sin\theta\cos\theta\cos^2\phi}{r}$$

$$+\frac{\partial f}{\partial r}\frac{\cos^2\theta\cos^2\phi}{r}+\frac{\partial^2 f}{\partial \theta^2}\frac{\cos^2\theta\cos^2\phi}{r^2}+\frac{\partial f}{\partial \theta}\frac{-\sin\theta\cos\theta\cos^2\phi}{r^2}$$

$$-\frac{\partial^2 f}{\partial \theta\partial \phi}\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}+\frac{\partial f}{\partial \phi}\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}$$

$$-\frac{\partial^2 f}{\partial \phi\partial r}\frac{\sin\phi\cos\phi}{r}+\frac{\partial f}{\partial r}\frac{\sin^2\phi}{r}-\frac{\partial^2 f}{\partial \phi\partial \theta}\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}$$

$$+\frac{\partial f}{\partial \theta}\frac{\cos\theta\sin^2\phi}{r^2\sin\theta}+\frac{\partial^2 f}{\partial \phi^2}\frac{\sin^2\phi}{r^2\sin^2\theta}+\frac{\partial f}{\partial \phi}\frac{\cos\phi\sin\phi}{r^2\sin^2\theta}$$

$$
\begin{aligned}
&= \frac{\partial^2 f}{\partial r^2} \sin^2\theta \cos^2\phi + \frac{\partial^2 f}{\partial r \partial \theta} \frac{2 \sin\theta \cos\theta \cos^2\phi}{r} - \frac{\partial f}{\partial \theta} \frac{2 \sin\theta \cos\theta \cos^2\phi}{r^2} \\
&\quad + \frac{\partial f}{\partial \theta} \frac{\cos\theta \sin^2\phi}{r^2 \sin\theta} - \frac{\partial^2 f}{\partial r \partial \phi} \frac{2 \cos\phi \sin\phi}{r} + \frac{\partial f}{\partial \phi} \frac{2 \sin\phi \cos\phi}{r^2 \sin^2\theta} \\
&\quad + \frac{\partial f}{\partial r} \frac{\cos^2\theta \cos^2\phi + \sin^2\phi}{r} + \frac{\partial^2 f}{\partial \theta^2} \frac{\cos^2\theta \cos^2\phi}{r^2} \\
&\quad - \frac{\partial^2 f}{\partial \phi \partial \theta} \frac{2 \cos\theta \cos\phi \sin\phi}{r^2 \sin\theta} + \frac{\partial^2 f}{\partial \phi^2} \frac{\sin^2\phi}{r^2 \sin^2\theta}
\end{aligned}
$$

y についても同様に

$$
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial y}\right)\frac{\partial \phi}{\partial y}
$$

$$
\begin{aligned}
&= \left\{ \frac{\partial^2 f}{\partial r^2} \sin\theta \sin\phi + \frac{\partial^2 f}{\partial r \partial \theta} \frac{\cos\theta \sin\phi}{r} + \frac{\partial f}{\partial \theta} \frac{-\cos\theta \sin\phi}{r^2} + \frac{\partial^2 f}{\partial r \partial \phi} \frac{\cos\phi}{r \sin\theta} \right. \\
&\quad \left. + \frac{\partial f}{\partial \phi} \frac{-\cos\phi}{r^2 \sin\theta} \right\} \sin\theta \sin\phi + \left\{ \frac{\partial^2 f}{\partial \theta \partial r} \sin\theta \sin\phi + \frac{\partial f}{\partial r} \cos\theta \sin\phi \right. \\
&\quad + \frac{\partial^2 f}{\partial \theta^2} \frac{\cos\theta \cos\phi}{r} + \frac{\partial f}{\partial \theta} \frac{-\sin\theta \sin\phi}{r} + \frac{\partial^2 f}{\partial \theta \partial \phi} \frac{\cos\phi}{r \sin\theta} \\
&\quad \left. + \frac{\partial f}{\partial \phi} \frac{-\cos\theta \cos\phi}{r \sin^2\theta} \right\} \frac{\cos\theta \sin\phi}{r} + \left\{ \frac{\partial^2 f}{\partial \phi \partial r} \sin\theta \sin\phi + \frac{\partial f}{\partial r} \sin\theta \cos\phi \right. \\
&\quad + \frac{\partial^2 f}{\partial \phi \partial \theta} \frac{\cos\theta \sin\phi}{r} + \frac{\partial f}{\partial \theta} \frac{\cos\theta \cos\phi}{r} + \frac{\partial^2 f}{\partial \phi^2} \frac{\cos\phi}{r \sin\theta} \\
&\quad \left. + \frac{\partial f}{\partial \phi} \frac{-\sin\phi}{r \sin\theta} \right\} \frac{\cos\phi}{r \sin\theta}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial^2 f}{\partial r^2} \sin^2\theta \sin^2\phi + \frac{\partial^2 f}{\partial r \partial \theta} \frac{2 \sin\theta \cos\theta \sin^2\phi}{r} - \frac{\partial f}{\partial \theta} \frac{2 \sin\theta \cos\theta \sin^2\phi}{r^2} \\
&\quad + \frac{\partial f}{\partial \theta} \frac{\cos\theta \cos^2\phi}{r^2 \sin\theta} + \frac{\partial^2 f}{\partial r \partial \phi} \frac{2 \cos\phi \sin\phi}{r} - \frac{\partial f}{\partial \phi} \frac{2 \sin\phi \cos\phi}{r^2 \sin^2\theta} \\
&\quad + \frac{\partial f}{\partial r} \frac{\cos^2\theta \sin^2\phi + \cos^2\phi}{r} + \frac{\partial^2 f}{\partial \theta^2} \frac{\cos^2\theta \sin^2\phi}{r^2} \\
&\quad + \frac{\partial^2 f}{\partial \phi \partial \theta} \frac{2 \cos\theta \cos\phi \sin\phi}{r^2 \sin\theta} + \frac{\partial^2 f}{\partial \phi^2} \frac{\cos^2\phi}{r^2 \sin^2\theta}
\end{aligned}
$$

z については

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right)\frac{\partial r}{\partial z}+\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial z}\right)\frac{\partial \theta}{\partial z}+\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial z}$$

$$=\left\{\frac{\partial^2 f}{\partial r^2}\cos\theta-\frac{\partial^2 f}{\partial r\partial \theta}\frac{\sin\theta}{r}-\frac{\partial f}{\partial \theta}\frac{-\sin\theta}{r^2}\right\}\cos\theta$$

$$+\left\{\frac{\partial^2 f}{\partial \theta\partial r}\cos\theta+\frac{\partial f}{\partial r}(-\sin\theta)-\frac{\partial^2 f}{\partial \theta^2}\frac{\sin\theta}{r}-\frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}\right\}\frac{-\sin\theta}{r}$$

$$=\frac{\partial^2 f}{\partial r^2}\cos^2\theta-\frac{\partial^2 f}{\partial r\partial \theta}\frac{2\sin\theta\cos\theta}{r}+\frac{\partial f}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}+\frac{\partial f}{\partial r}\frac{\sin^2\theta}{r}$$

$$+\frac{\partial^2 f}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

結果を加えるとラプラシアンは次のように書ける

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{\partial^2 f}{\partial r^2}(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta)$$

$$+\frac{\partial^2 f}{\partial r\partial \theta}\frac{2\sin\theta\cos\theta\cos^2\phi+2\sin\theta\cos\theta\sin^2\phi-2\sin\theta\cos\theta}{r}$$

$$+\frac{\partial f}{\partial \theta}\frac{-2\sin\theta\cos\theta\cos^2\phi-2\sin\theta\cos\theta\sin^2\phi+2\sin\theta\cos\theta}{r^2}$$

$$+\frac{\partial f}{\partial \theta}\frac{\cos\theta\sin^2\phi+\cos\theta\cos^2\phi}{r^2\sin\theta}+\frac{\partial^2 f}{\partial r\partial \phi}\frac{-2\cos\phi\sin\phi+2\cos\phi\sin\phi}{r}$$

$$+\frac{\partial f}{\partial \phi}\frac{2\sin\phi\cos\phi-2\sin\phi\cos\phi}{r^2\sin^2\theta}$$

$$+\frac{\partial f}{\partial r}\frac{\cos^2\theta\cos^2\phi+\sin^2\phi+\cos^2\theta\sin^2\phi+\cos^2\phi+\sin^2\theta}{r}$$

$$+\frac{\partial^2 f}{\partial \theta^2}\frac{\cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta}{r^2}$$

$$+\frac{\partial^2 f}{\partial \phi\partial \theta}\frac{-2\cos\theta\cos\phi\sin\phi+2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}+\frac{\partial^2 f}{\partial \phi^2}\frac{\sin^2\phi+\cos^2\phi}{r^2\sin^2\theta}$$

$$=\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

連立１次方程式を解くのに行列式を使用したが変数の少ない表示を用いると直接解ける。
ここでも基本は偏微分の連鎖則でこれがきちんと理解できていれば容易である。
３次元の極座標変換を求めてみよう。次の３式をもとにする。

$$r^2 = x^2 + y^2 + z^2 \qquad (4)$$

$$tan\ \phi = y / x \qquad (5)$$

$$cos\ \theta = z / r \qquad (6)$$

(4) の両辺をｘで偏微分する（ｒだけの関数に注意θ，φによらない）

$$\frac{\partial}{\partial x} r^2 = \frac{\partial}{\partial r} r^2 \frac{\partial r}{\partial x} = 2r \frac{\partial r}{\partial x} \ , \quad \frac{\partial}{\partial x}\left(x^2 + y^2 + y^2\right) = 2x ,$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r \sin\theta \cos\phi}{r} = \sin\ \theta\ \cos\ \phi$$

y, z でも同様に偏微分すると。次の関係式が得られる。

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\ \theta\ \sin\ \phi \ , \qquad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\ \theta$$

次に (5) 式の両辺をｘで偏微分する（φだけの関数に注意ｒ，θによらない）

$$\frac{\partial}{\partial x} \tan\ \phi = \frac{\partial}{\partial \phi} \tan\ \phi \frac{\partial \phi}{\partial x} = \frac{1}{\cos^2\phi} \frac{\partial \phi}{\partial x} \ , \qquad \frac{\partial}{\partial x} \frac{y}{x} = \frac{-y}{x^2}$$

$$\frac{\partial \phi}{\partial x} = \cos^2 \phi \frac{-y}{x^2} = \cos^2\phi \frac{-\ r \sin\theta \sin\phi}{(r \sin\theta \cos\phi)^2} == \frac{-\ \sin\phi}{r \sin\theta}$$

y, z でも同様に偏微分すると。次の関係式が得られる。

$$\frac{\partial \phi}{\partial y} = \cos^2 \phi \frac{1}{x} = \frac{\cos\phi}{r \sin\theta} \ , \qquad \frac{\partial \phi}{\partial z} = 0$$

最後に (6) 式の両辺をｘで偏微分する（ｒとθの関数に注意）次いでｙ，ｚで偏微分して

$$\frac{\partial}{\partial x} \cos\theta = -\sin\theta \frac{\partial \theta}{\partial x} \ , \quad \frac{\partial}{\partial x} \frac{z}{r} = \frac{\partial}{\partial r} \frac{z}{r} \frac{\partial r}{\partial x} = \frac{-\ z}{r^2} \frac{\partial r}{\partial x} = \frac{-\ \cos\theta}{r} \frac{x}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{-\cos\theta}{r(-\sin\theta)} \sin\theta \cos\phi = \frac{\cos\theta \cos\phi}{r}$$

$$\frac{\partial}{\partial y} \cos\theta = -\sin\theta \frac{\partial \theta}{\partial y} = \frac{-\ z}{r^2} \frac{\partial r}{\partial y} = \frac{-\ \cos\theta}{r} \sin\theta \sin\phi \ , \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta \sin\phi}{r}$$

$$\frac{\partial}{\partial z} \cos\theta = -\sin\theta \frac{\partial \theta}{\partial z} = \frac{-\ z}{r^2} \frac{\partial r}{\partial z} + \frac{1}{r} = \frac{1}{r} (1 - \cos^2\theta), \qquad \frac{\partial \theta}{\partial z} = -\ \frac{\sin\theta}{r}$$

が得られる。